K_1 OF SOME ABELIAN CATEGORIES

BY
LESLIE G. ROBERTS

1. **Introduction.** Let $\mathscr{A}$ be an abelian category. Then the groups $K_0(\mathscr{A})$ and $K_1(\mathscr{A})$ have been defined in [2] or [3]. I will recall their definition.

Let $F(\mathscr{A})$ be the free abelian group on isomorphism classes of objects of $\mathscr{A}$ and let (A) be the basis element of $F(\mathscr{A})$ corresponding to the isomorphism class of the object A of $\mathscr{A}$. Then $K_0(\mathscr{A})$ is $F(\mathscr{A})$ factored out by the subgroup of $F(\mathscr{A})$ generated by all $(A)-(A_1)-(A_2)$ where $0 \to A_1 \to A \to A_2 \to 0$ is an exact sequence in $\mathscr{A}$.

Denote by $\kappa_0(A)$ the image of (A) in $K_0(\mathscr{A})$.

In order to define $K_1(\mathscr{A})$ it is convenient to first define a new category $\Omega\mathscr{A}$. The objects of $\Omega\mathscr{A}$ are pairs (A, α) where A is an object of $\mathscr{A}$ and α is an automorphism of A. A morphism $(A_1, \alpha_1) \to (A_2, \alpha_2)$ consists of a morphism $f: A_1 \to A_2$ such that $\alpha_2 f = f\alpha_1$.

It is easily seen that $\Omega\mathscr{A}$ is abelian. A sequence $(A_1, \alpha_1) \xrightarrow{f} (A_2, \alpha_2) \xrightarrow{g} (A_3, \alpha_3)$ in $\Omega\mathscr{A}$ is exact if and only if the corresponding sequence $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ is exact in $\mathscr{A}$.

Then $K_1(\mathscr{A})$ is equal to $K_0(\Omega\mathscr{A})$ factored out by the subgroup generated by all $\kappa_0(A, \alpha\beta)-\kappa_0(A, \alpha)-\kappa_0(A, \beta)$ where $A \in \operatorname{obj}(\mathscr{A})$ and α, β are automorphism of A.

Denote by $\kappa_1(A, \alpha)$ the image of $\kappa_0(A, \alpha)$ in $K_1(\mathscr{A})$.

Now assume that $\mathscr{A}$ satisfies the following two conditions:

(i) For all objects A, B of $\mathscr{A}$, $\mathscr{A}(A, B)$ is a finite dimensional vector space over an algebraically closed field k. ($\mathscr{A}(A, B)$ denotes the morphisms in $\mathscr{A}$ from A to B.)

(ii) For all maps f, g in A, $f \in \mathscr{A}(A', A)$, $g \in \mathscr{A}(B, B')$ $\mathscr{A}(f, g): \mathscr{A}(A, B) \to \mathscr{A}(A', B')$ is a k-homomorphism.

The aim of this paper is to prove that under these conditions

$$K_0(\mathscr{A}) \otimes_Z k^* \cong K_1(\mathscr{A}).$$

This paper is based on part of the author's doctoral dissertation at Harvard University under the direction of Professor D. Mumford. A number of suggestions of the referee have been adopted.

2. **Definition of Φ.** If A is a nonzero object of $\mathscr{A}$, then $\mathscr{A}(A, A)$ is nonzero, and condition (i) implies that $\mathscr{A}(A, A)$ contains a nonzero copy of k, i.e. $k \cdot 1_A$. I will omit the 1_A for convenience of notation.

Received by the editors March 27, 1968 and, in revised form, May 16, 1968.

Condition (ii) implies that if $f \in \mathscr{A}(A, B)$ then $f\lambda = \lambda f$, $\lambda \in k$. In particular, taking $A = B$, we see that k lies in the center of $\mathscr{A}(A, A)$.

If $\lambda \in k^*$, then λ is an automorphism of A. Define a map

$$\mathrm{obj}\,(\mathscr{A}) \times k^* \to K_1(\mathscr{A})$$

by $(A, \lambda) \to \kappa_1(A, \lambda)$. It is easily seen that this defines a homomorphism

$$\Phi\colon K_0(\mathscr{A}) \otimes k^* \to K_1(\mathscr{A})$$

with $\Phi(\kappa_0(A) \otimes \lambda) = \kappa_1(A, \lambda)$. (All tensor products will be over the integers, so I will omit the Z.) I will show that Φ is an isomorphism.

3. **The functors F_λ.** Let A be a nonzero object of $\mathscr{A}$ and let $\alpha \in \mathscr{A}(A, A)$. Consider the k-algebra homomorphism $k[X] \to \mathscr{A}(A, A)$ given by $X \to \alpha$. Since $\mathscr{A}(A, A)$ is a finite dimensional vector space, this homomorphism has nontrivial kernel (f). Then $f \in k[X]$ is the polynomial of lowest degree satisfied by α. Take f to be monic for definiteness. Since k is algebraically closed, we have

$$f(X) = \prod_{i=1}^{r} (X-\lambda_i)^{n_i},$$

with $\sum_{i=1}^{r} n_i = n$, $\lambda_i \neq \lambda_j$, if $i \neq j$. The λ_i will be called eigenvalues of α, and f will be called the minimal equation for α. α is an automorphism if and only if no λ_i is zero.

We have

$$k[\alpha] \cong \frac{k[X]}{(f)} \cong \frac{k[X]}{(X-\lambda_1)^{n_1}} \oplus \frac{k[X]}{(X-\lambda_2)^{n_2}} \oplus \cdots \oplus \frac{k[X]}{(X-\lambda_r)^{n_r}}.$$

Let $p_i(\alpha)$, $1 \leqq i \leqq r$, be idempotents in the above direct sum decomposition. Then we have a direct sum decomposition $A \cong A_1 \oplus A_2 \oplus \cdots \oplus A_r$ where $A_i = \mathrm{im}\, p_i(\alpha)$.

It is easy to see that this direct sum decomposition is stable under α, and that $A_i = \ker\,(\alpha-\lambda_i)^{n_i} =$ the subobject of A on which $\alpha - \lambda_i$ is nilpotent.

I will summarize the properties of this decomposition in the following theorem:

THEOREM 1. *Let α be an endomorphism of $A \in \mathrm{obj}\,(\mathscr{A})$, $A \neq 0$. If*

$$f(X) = \prod_{i=1}^{r} (X-\lambda_i)^{n_i} \qquad (\lambda_i \neq \lambda_j \text{ if } i \neq j)$$

is the minimal equation of α, then $A \cong \bigoplus_{i=1}^{r} \ker\,(\alpha-\lambda_i)^{n_i} = \bigoplus_{i=1}^{r} A_i$. The A_i have the following properties: (a) *A_i is stable under α.* (b) *α restricted to A_i has one eigenvalue λ_i ($\lambda_i \neq \lambda_j$ if $i = j$). Furthermore any direct sum decomposition with properties* (a) *and* (b) *is isomorphic to $\bigoplus_{i=1}^{r} A_i$.*

The uniqueness assertion is easy to prove.

COROLLARY. *If M is indecomposable, every endomorphism of M is an automorphism or nilpotent, and $\mathscr{A}(M, M)$ is a local ring.*

Now suppose that $(A, \alpha) \in \Omega\mathscr{A}$. For all $\lambda \in k^*$, let $(A, \alpha)_\lambda$ be the subobject of A on which $\alpha - \lambda$ is nilpotent. Let α_λ be the restriction of α to $(A, \alpha)_\lambda$. Then $((A, \alpha)_\lambda, \alpha_\lambda) \in \Omega\mathscr{A}$. If $(A, \alpha) \xrightarrow{f} (B, \beta)$ is a morphism, then $\beta f = f\alpha$. Hence also $(\beta - \lambda)^n f = f(\alpha - \lambda)^n$ for all n. Therefore f maps $(A, \alpha)_\lambda$ into $(B, \beta)_\lambda$. Hence, for all $\lambda \in k^*$, we have a functor $F_\lambda \colon \Omega\mathscr{A} \to \Omega\mathscr{A}$ which sends $(A, \alpha) \to ((A, \alpha)_\lambda, \alpha_\lambda)$.

THEOREM 2. *If* $(A, \alpha) \in \operatorname{obj} \Omega\mathscr{A}$, *then*

$$(A, \alpha) \cong \bigoplus_{\lambda \in k^*} ((A, \alpha)_\lambda, \alpha_\lambda)$$

and the functors F_λ *are exact*, $\lambda \in k^*$.

Proof. The first assertion follows from Theorem 1. (There are only a finite number of nonzero $(A, \alpha)_\lambda$ so the direct sum makes sense.)

To prove the exactness, let

$$0 \longrightarrow (A_1, \alpha_1) \xrightarrow{f} (A, \alpha) \xrightarrow{g} (A_2, \alpha_2) \longrightarrow 0$$

be an exact sequence in $\Omega\mathscr{A}$. Then for all $\lambda \in k^*$ we get a sequence S_λ:

$$0 \to F_\lambda(A_1, \alpha_1) \to F_\lambda(A, \alpha) \to F_\lambda(A_2, \alpha_2) \to 0.$$

If we take the direct sum of these sequences over all λ we get the original exact sequence back again. Therefore S_λ must be an exact sequence for all λ, and hence the functors F_λ are exact.

4. **Proof that Φ is surjective.** Let α be an automorphism of $A \in \operatorname{obj} \mathscr{A}$. Then $(A, \alpha) \cong \bigoplus_{\lambda \in k^*} ((A, \alpha)_\lambda, \alpha_\lambda)$ by Theorem 2. Thus $\kappa_1(A, \alpha) = \sum_{\lambda \in k^*} \kappa_1((A, \alpha)_\lambda, \alpha_\lambda)$. Thus we are reduced to showing that $\kappa_1(A, \alpha)$ lies in the image of Φ, where α has one eigenvalue $\lambda \in k^*$, i.e. $(\alpha - \lambda)^n = 0$ for some $n > 0$.

Then we have a filtration of A by subobjects

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = A,$$

where $A_i = \ker (\alpha - \lambda)^i$, $0 \leqq i \leqq n$, and this filtration is stable under α. Since $(\alpha - \lambda) A_i \subset A_{i-1}$, $1 \leqq i \leqq n$, α must induce scalar multiplication by λ on the quotient object A_i/A_{i-1}, $1 \leqq i \leqq n$. Therefore $\kappa_1(A, \alpha) = \sum_{i=1}^n \kappa_1(A_i/A_{i-1}, \lambda) = \kappa_1(A, \lambda)$ lies in the image of Φ, thus proving that Φ is surjective.

5. **The radical of $\mathscr{A}$.** My aim is to define an inverse to Φ. I will first discuss the radical of $\mathscr{A}$.

The Krull-Schmidt theorem holds in an abelian category satisfying conditions (i) and (ii) of the introduction, by Atiyah [1]. That is, every object in $\mathscr{A}$ can be expressed uniquely as the direct sum of a finite number of indecomposable objects.

Furthermore, if M is an indecomposable object in $\mathscr{A}$, then $\mathscr{A}(M, M)$ is a local ring by the corollary to Theorem 1. Let m be the radical of $\mathscr{A}(M, M)$. Then $\mathscr{A}(M, M)/m$ is a division ring which contains k in its center. Since k is assumed to be algebraically closed we must have $\mathscr{A}(M, M)/m = k$.

Also we have

LEMMA 1. *If A and B are nonisomorphic indecomposable objects, then any composition $A \xrightarrow{f} B \xrightarrow{g} A$ lies in the radical of $\mathscr{A}(A, A)$.*

Proof. Otherwise gf would be an isomorphism. This would make A a direct summand of B, which is impossible.

An ideal $\mathscr{I}$ of the category $\mathscr{A}$ consists of a subgroup $\mathscr{I}(A, B) \subset \mathscr{A}(A, B)$ $A, B \in \operatorname{obj} \mathscr{A}$, such that if $f \in \mathscr{I}(A, B)$, $g \in \mathscr{A}(B, C)$ and $h \in \mathscr{A}(D, A)$, then $gfh \in \mathscr{I}(D, C)$.

The radical of an additive category is defined in Kelley [4]. It is defined to be the unique ideal $\mathscr{R}$ in $\mathscr{A}$ such that $\mathscr{R}(A, A)$ is the radical of $\mathscr{A}(A, A)$.

If A and B are objects of $\mathscr{A}$, then we may write each as the direct sum of a finite number of indecomposable objects, $A \cong \bigoplus_{i=1}^{n} A_i$, $B \cong \bigoplus_{j=1}^{m} B_j$. Then any morphism $f \in \mathscr{A}(A, B)$ will be given by an $m \times n$ matrix with entries f_{ij} in $\mathscr{A}(A_j, B_i)$. Then by Lemma 1 above, and Lemmas 1 to 6 of Kelley [4] $\mathscr{R}(A, B)$ consists of those f such that no f_{ij} is an isomorphism.

Define the category $\mathscr{A}/\mathscr{R}$ by letting the objects of $\mathscr{A}/\mathscr{R}$ be the same as those of $\mathscr{A}$, and $\mathscr{A}/\mathscr{R}(A, B) = \mathscr{A}(A, B)/\mathscr{R}(A, B)$, and let N be the canonical functor $\mathscr{A} \to \mathscr{A}/\mathscr{R}$. Then N is additive, and α is an isomorphism if and only if $N\alpha$ is an isomorphism.

Let $\{A_i\}_{i \in I}$ be a representative set of indecomposable objects of $\mathscr{A}$. Then $\mathscr{A}/\mathscr{R}(A_i, A_i) = k$, $\forall i$, and $\mathscr{A}/\mathscr{R}(A_i, A_j) = 0$, $\forall i, j$, $i \neq j$. Let $\mathscr{V}$ be the category whose objects consist of k^n $\forall n \geqq 0$, and whose morphisms are all vector space homomorphisms. Let $\coprod_{i \in I} \mathscr{V}$ be the subcategory of the product such that all but a finite number of the coordinates are zero. Define a functor

$$G\colon \coprod_{i \in I} \mathscr{V} \to \mathscr{A}$$

by setting $G(k^n, i) = \bigoplus_{j=1}^{n} A_i$. If $\alpha\colon k^n \to k^m$, let $G(\alpha, i)\colon \bigoplus_{j=1}^{n} A_i \to \bigoplus_{j=1}^{m} A_i$ be given by the same matrix.

Then from the description of morphisms in $\mathscr{A}/\mathscr{R}$ given above it is easily seen that NG gives an equivalence of categories.

Thus $\mathscr{A}/\mathscr{R}$ is abelian, and $K_1(\mathscr{A}/\mathscr{R}) = \bigoplus_{i \in I} k^*$, the map $\kappa_1\colon F(\Omega\mathscr{V}) \to k^*$ being given by the determinant.

6. **Definition of the map Ψ.** We can now define a homomorphism $\Gamma\colon K_1(\mathscr{A}/\mathscr{R}) \to K_0(\mathscr{A}) \otimes k^*$ by $\Gamma(\lambda, i) = \kappa_0(A_i) \otimes \lambda$. Let $(A, \alpha) \in \operatorname{obj}(\Omega\mathscr{A})$. Define $\Psi(A, \alpha) = \Gamma\kappa_1(NA, N\alpha)$. Then I claim that Ψ defines a homomorphism

$$K_1(\mathscr{A}) \to K_0(\mathscr{A}) \otimes k^*.$$

First I will check that Ψ vanishes on the defining relations for $K_0(\Omega\mathscr{A})$. Let $(A, \alpha) \in \Omega\mathscr{A}$. Then $(A, \alpha) \cong \bigoplus_{\lambda \in k^*} ((A, \alpha)_\lambda, \alpha_\lambda)$. Therefore

$$\kappa_1(N(A, \alpha)) = \sum_{\lambda \in k} \kappa_1(N((A, \alpha)_\lambda), N\alpha_\lambda).$$

I claim that $\kappa_1(N((A, \alpha)_\lambda), N\alpha_\lambda) = \kappa_1(N((A, \alpha)_\lambda), \lambda)$. Then it will follow immediately from the exactness of the functors F_λ that Ψ vanishes on the defining relations for $K_0(\Omega\mathscr{A})$. This will follow from

LEMMA 2. *Let α be an automorphism of A such that $(\alpha-\lambda)^n=0$ $(n>0)$. Then $\kappa_1(NA, N\alpha)=\kappa_1(NA, \lambda)$.*

Proof. $(\alpha-\lambda)^n=0$. Therefore $(N\alpha-\lambda)^n=0$. Therefore the matrix $N\alpha$ has one eigenvalue λ. This can be checked by using the Jordan Canonical form for α. Hence if $A \cong \bigoplus n_i A_i$ then $\kappa_1(NA, N\alpha) = \bigoplus_i \lambda^{n_i} = \kappa_1(NA, \lambda)$ as required.

This proves that Ψ vanishes on the defining relations for $K_0(\Omega\mathscr{A})$. Now I will check that Ψ vanishes on the other relation defining $K_1(\mathscr{A})$.

If α and β are automorphisms of A, then

$$\begin{aligned}\Psi(A, \alpha\beta) &= \Gamma\kappa_1(NA, N(\alpha\beta)) = \Gamma\kappa_1(NA, (N\alpha)(N\beta)) \\ &= \Gamma(\kappa_1(NA, N\alpha)+\kappa_1(NA, N\beta)) \\ &= \Psi(A, \alpha)+\Psi(A, \beta)\end{aligned}$$

as required.

Therefore Ψ defines a homomorphism from $K_1(\mathscr{A}) \to K_0(\mathscr{A}) \otimes k^*$ which I will also denote by Ψ.

7. **Proof that Φ is an isomorphism.** Φ has already been shown to be onto. We will now show that $\Psi\Phi: K_0(\mathscr{A}) \otimes k^* \to K_0(\mathscr{A}) \otimes k^*$ equals the identity. Then Φ will be a monomorphism also.

$$\Phi(\kappa_0(A) \otimes \lambda) = \kappa_1(A, \lambda), \quad \Psi(\kappa_1(A, \lambda)) = \kappa_0(A) \otimes \lambda.$$

Therefore $\Psi\Phi = $ identity, and Φ is an isomorphism.

Thus my results may be stated as follows:

THEOREM 3. *Let $\mathscr{A}$ be an abelian category which satisfies conditions* (i) *and* (ii) *of the introduction. Then the map $\Phi: K_0(\mathscr{A}) \otimes k^* \to K_1(\mathscr{A})$ defined by $\Phi(\kappa_0(A) \otimes \lambda) = \kappa_1(A, \lambda)$ is an isomorphism.*

8. **Examples.** Some examples of abelian categories which satisfy conditions (i) and (ii) of the introduction are the following:

(1) The category $\mathscr{C}$ of coherent sheaves on a projective algebraic variety X, over an algebraically closed field k. In particular, if X is a nonsingular curve, then

$$K_0(\mathscr{C}) = Z \oplus \operatorname{Pic}(X) = Z \oplus Z \oplus \operatorname{Pic}_0(X),$$

and so

$$K_1(\mathscr{C}) = k^* \oplus k^* \oplus (\operatorname{Pic}_0(X) \otimes k^*).$$

(2) The category of left modules of finite type over a finite dimensional k-algebra.

I will conclude by giving a couple of examples where all hypotheses are satisfied except the algebraic closure of the field k. Suppose that k is of finite index in its algebraic closure $\bar{k}$.

(3) Let $\mathscr{A}$ be an abelian category which satisfies the hypotheses of Theorem 3 over $\bar{k}$. Then the hypotheses are also satisfied over k (except for algebraic closure of k). But $K_1(\mathscr{A}) \cong K_0(\mathscr{A}) \otimes \bar{k}^*$.

(4) Let $\mathscr{V}$ be the category of finite dimensional vector spaces over k. In this case $K_1(\mathscr{V}) = k^* = K_0(\mathscr{V}) \otimes k^*$.

References

1. M. F. Atiyah, *On the Krull-Schmidt theorem with application to sheaves*, Bull. Soc. Math. France **84** (1956), 307–317.

2. H. Bass, A. Heller and R. Swan, *The Whitehead group a polynomial extension*, Inst. Hautes Etudes Sci. Publ. Math., No. 22, 1964, pp. 61–79.

3. A. Heller, *Some exact sequences in algebraic K-theory*, Topology **4** (1965), 389–408.

4. G. M. Kelley, *The radical of a category* J. Austral. Math. Soc. **4** (1964), 299.

Harvard University,
Cambridge, Massachusetts